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Optical Solitons with Time-Dependent Dispersion, Nonlinearity and Attenuation in a Kerr-Law Media

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Abstract This paper obtains the 1-soliton solution of the nonlinear Schrödinger's equation with Kerr law nonlinearity and time-dependent dispersion, nonlinearity and attenuation. The solitary wave ansatze is used to obtain this solution. The constraint relation between these time-dependent coefficients is also obtained for the solitons to exist. The variation of the soliton velocity also falls out by this method.

Keywords Optical solitons · Kerr law nonlinearity · Attenuation · Integrability

1 Introduction

The study of optical solitons in a Kerr law media is an important area of study. It is governed by the nonlinear Schrodinger's equation (NLSE) [1–10]. It studies the propagation of solitons through optical fibers for trans-continental and trans-oceanic distances. Thus the dynamics of solitons governed by the NLSE is well understood and well known in this context. The perturbation terms are also very well studied in this context and research in this area is flooded with various papers.

Recently, some attention is being paid to the NLSE with time-dependent coefficients. In such a situation the NLSE is not integrable by the classical method of integrability, namely the Inverse Scattering Transform (IST). This technique is used for integrating nonlinear evolution equations and the NLSE is one such equation. IST is the nonlinear analog of Fourier transform that is used to solve linear partial differential equations. Thus, as mentioned, when it comes to the case of NLSE with time-dependent coefficients, the method of IST does not work since the corresponding Painleve test of integrability fails [9].

Thanks to the various recent techniques of integrability that are used to integrate a wide variety of nonlinear evolution equations. These methods include the Wadati trace method, variational iteration method, Riccati equation expansion method, pseudo-spectral method,

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sine-cosine method, tanh-sech method and many more. While the advantage of these modern methods integrability is that they can obtain periodic, double periodic solutions, rational solutions and the soliton solutions to these nonlinear evolution equations, the disadvantage of these methods as compared to the IST technique is that it cannot compute the soliton radiation nor can it lay down the conserved quantities of these equations. But nevertheless, it makes a big difference when a soliton solution is obtained. In this paper, the solitary wave ansatze will be used to obtain the 1-soliton solution of the NLSE with time-dependent dispersion, nonlinearity and attenuation.

2 Mathematical Analysis

The dimensionless form of the NLSE with time-dependent dispersion, nonlinearity and attenuation is given by

$$iq_t + \frac{a(t)}{2} q_{xx} + b(t)|q|^2 q = i\alpha(t) q \tag{1}$$

In (1), the first term represents the evolution term, while the second and third terms respectively represent the dispersion and nonlinear terms. The right hand side represent the attenuation term. The type of nonlinearity that is seen in (1) is known as the Kerr law nonlinearity. The dispersion term and the nonlinear term has time dependent coefficients given by $a(t)$ and $b(t)$ respectively while the coefficient of the attenuation term is given by $\alpha(t)$. The solitons are a result of a balance between the dispersion and nonlinearity.

It is well known that the solitons due to NLSE with Kerr law nonlinearity is given by the sech function [1–10]. Therefore, without any loss of generality, one can assume that the 1-soliton solution of (1) is given by

$$q(x,t) = \frac{A}{\cosh[B(x - vt)]} e^{i(-\kappa x + \omega t + \theta)} \tag{2}$$

where A is the amplitude of the soliton, B is the inverse width and v is the soliton velocity. Also, κ represents the frequency, ω is the wave number while θ is the phase. It is to be noted that since in (1), the coefficients of dispersion, nonlinearity and attenuation are time-dependent and not constants, the soliton parameters A, B, κ, ω and θ are time dependent, namely $A = A(t)$, $B = B(t)$, $\kappa = \kappa(t)$, $\omega = \omega(t)$ and $\theta = \theta(t)$. Thus, from (2),

$$\begin{aligned} q_t = \Bigg[& \frac{dA}{dt} \frac{1}{\cosh\tau} \\ & - A \frac{\tanh\tau}{\cosh\tau} \left\{ \frac{\tau}{B} \frac{dB}{dt} - B\left(v + t\frac{dv}{dt} \right) \right\} + iA\left(-x\frac{d\kappa}{dt} + \omega + t\frac{d\omega}{dt} + \frac{d\theta}{dt} \right) \Bigg] e^{i\phi} \end{aligned} \tag{3}$$

$$q_{xx} = \left[AB^2 \left(\frac{1}{\cosh\tau} - \frac{2}{\cosh^3\tau} \right) + 2i\kappa AB \frac{\tanh\tau}{\cosh\tau} - \kappa^2 A \frac{1}{\cosh\tau} \right] e^{i\phi} \tag{4}$$

where

$$\tau = B(x - vt) \tag{5}$$

and

$$\phi = -\kappa x + \omega t + \theta \tag{6}$$

Substituting (4) and (5) into (1) and equating the real and imaginary parts leads to the following pair of relations respectively

$$\frac{dA}{dt}\frac{1}{\cosh\tau} - A\frac{\tanh\tau}{\cosh\tau}\left\{\frac{\tau}{B}\frac{dB}{dt} - B\left(v + t\frac{dv}{dt}\right)\right\} + a(t)\kappa AB\frac{\tanh\tau}{\cosh\tau} = \alpha(t)\frac{A}{\cosh\tau} \tag{7}$$

and

$$-\frac{A}{\cosh\tau}\left(-x\frac{d\kappa}{dt} + \omega + t\frac{d\omega}{dt} + \frac{d\theta}{dt}\right) + \frac{a(t)AB^2}{2}\left(\frac{1}{\cosh\tau} - \frac{2}{\cosh^3\tau}\right) - \frac{a(t)\kappa^2 A}{2\cosh\tau} + \frac{b(t)A^3}{\cosh^3\tau} = 0 \tag{8}$$

From (7), equating the coefficients of $1/\cosh\tau$, one gets

$$\frac{dA}{dt} = \alpha(t)A \tag{9}$$

which gives

$$A(t) = A_0 e^{\int \alpha(t)dt} \tag{10}$$

where A_0 is the initial amplitude of the soliton. Again from (7), setting the coefficient of $\tau\cosh\tau/\tanh\tau$ to zero yields

$$\frac{dB}{dt} = 0 \tag{11}$$

which shows that the inverse width of the soliton $B(t)$ must be a constant. Again, from (7), setting the coefficient of $\tanh\tau/\cosh\tau$ to zero yields

$$v(t) = -\frac{\int \kappa a(t)dt}{t} \tag{12}$$

From (8), equating the coefficients of $1/\cosh^3\tau$ leads to

$$a(t)B^2 = b(t)A^2 \tag{13}$$

which, by virtue of (10), leads to

$$B(t) = A_0 e^{\int \alpha(t)dt}\sqrt{\frac{b(t)}{a(t)}} \tag{14}$$

From (14), one needs to have $a(t)b(t) > 0$. Now from (11), for $B(t)$ to be a constant, (14) yields

$$b(t)e^{2\int \alpha(t)dt} = ka(t) \tag{15}$$

where $k \in R$ is a constant. Finally, in (8), setting the coefficient of $1/\cosh\tau$ to zero gives

$$\omega(t) = \frac{a(t)}{2}(B^2 - \kappa^2) \tag{16}$$

and

$$\frac{d\kappa}{dt} = 0 \tag{17}$$

$$\frac{d\theta}{dt} = 0 \tag{18}$$

so that

$$\theta = \theta_0 \tag{19}$$

where, θ_0, is a constant. Thus, the velocity of the soliton can be written as

$$v(t) = -\frac{\kappa}{t}\int a(t)dt \tag{20}$$

Thus, finally the 1-soliton solution of the NLSE (1) is given by (2), where the amplitude $A(t)$ is given by (9) and the width $B(t)$ stays constant. This compels the constraint between the time-dependent coefficients given by (15) which therefore serves as a condition for the solitons to exist. The velocity $v(t)$ is given by (20) while the other soliton parameters namely κ, ω and θ all stay constant. The only other simple condition that needs to hold for the solitons to exist is that $a(t)$ and $\alpha(t)$ are to be Riemann integrable and that $a(t)b(t) > 0$ which follows from (14) and (20).

3 Conclusions

In this paper, the solitary wave ansatze is used to obtain the 1-soliton solution of the NLSE with time-dependent dispersion, nonlinearity and attenuation. It is only necessary that these time-dependent coefficients be Riemann integrable. Also, the constraint relation between these coefficients is obtained. The method that is used is far less involved than the standard techniques that are used to study these kind of problems. In future, this method will be extended to obtain the 1-soliton solution in presence of additional terms like the Raman scattering, higher order dispersions, self-steepening and many more.

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References

1. Chow, K.W., Lam, C.K., Nakeeran, K., Malomed, B.: Transmission and stability of solitary pulses in complex Ginzburg–Landau equations with variable coefficients. J. Phys. Soc. Jpn. **77**(5), 054001 (2008)
2. Geng, Y., Li, J.: Exact solutions to a nonlinearly dispersive Schrödinger equation. Appl. Math. Comput. **195**(2), 420–439 (2008)
3. Konar, S., Mishra, M., Jana, S.: Nonlinear evolution of cosh-Gaussian laser beams and generations of flat top solitons in cubic-quintic nonlinear media. Phys. Lett. A **362**(5–6), 505–510 (2007)
4. Liu, J.-G., Li, Y.-Z., Tian, B.: Soliton-like solutions for the modified variable-coefficient Ginzburg–Landau equation. Commun. Nonlinear Sci. Numer. Simul. (to appear)
5. Lü, X., Zhu, H.-W., Meng, X.-H., Yang, Z.-C., Tian, B.: Soliton solutions and a Bäcklund transformation for a generalized nonlinear Schrödinger equation with variable coefficients from optical fiber communications. J. Math. Anal. Appl. **336**, 1305–1315 (2007)

6. Lü, X., Zhu, H.-W., Yao, Z.-Z., Meng, X.-H., Zhang, C., Zhang, C.-Y., Tian, B.: Multisoliton solutions in terms of double Wronskian determinant for a generalized variable-coefficient nonlinear Schödinger equation from plasma physics, arterial mechanics, fluid dynamics and optical communications. Ann. Phys. **323**(8), 1947–1955 (2008)
7. Tian, B., Gao, Y.-T.: Symbolic-computation study of the perturbed nonlinear Schrödinger model in inhomogeneous optical fibers. Phys. Lett. A **342**, 228–236 (2005)
8. Xu, T., Li, J., Zhang, H.-Q., Zhang, Y.-X., Hu, W., Gao, Y.-T., Tian, B.: Integrable aspects and applications of a generalized inhomogeneous N-coupled nonlinear Schrödinger system in plasmas and optical fibers via symbolic computation. Phys. Lett. A **372**(12), 1990–2001 (2008)
9. Zhidkov, P.E.: Korteweg-de Vries and Nonlinear Schrödinger's Equations: Qualitative Theory. Springer, New York (2001)
10. Zu, H.-W., Tian, B.: Bäcklund transformation in bilinear form for a higher-order nonlinear Schrödinger equation. Nonlinear Anal.: Theory, Methods Appl. (to appear)